If the buckling problem is considered, the following state equation can be derived from the basic equations

$$\frac{\partial}{\partial z}\begin{Bmatrix}\sigma_z\\ u\\ w\\ \tau_{xz}\end{Bmatrix} = \begin{bmatrix} 0 & 0 & \dfrac{p}{1+w^0_{,z}}\dfrac{\partial^2}{\partial x^2} & -\dfrac{\partial}{\partial x}\\ 0 & 0 & -\dfrac{\partial}{\partial x} & \dfrac{1}{C_{55}}\\ \dfrac{1}{C_{33}} & -\dfrac{C_{13}}{C_{33}}\dfrac{\partial}{\partial x} & 0 & 0\\ -\dfrac{C_{13}}{C_{33}}\dfrac{\partial}{\partial x} & \left(\dfrac{p}{1+u^0_{,x}}-\alpha\right)\dfrac{\partial^2}{\partial x^2} & 0 & 0 \end{bmatrix}\begin{Bmatrix}\sigma_z\\ u\\ w\\ \tau_{xz}\end{Bmatrix} \tag{12}$$

where p is the uniform compressive load applied at the two ends along the axial direction and

$$u^0_{,x} = -\frac{C_{33}p}{C_{11}C_{33}-C_{13}^2}, \qquad w^0_{,z} = \frac{C_{13}p}{C_{11}C_{33}-C_{13}^2} \tag{13}$$

are the normal strains in the prebuckling state. The buckling analysis for the nondebonded imperfect beams is then similar to that presented earlier for the free-vibration problem. In the case of a generally debonded beam, an exact buckling solution cannot be obtained and approximate treatments should be employed. However, if the beam is with through-length delamination at the kth interface, for example, an exact solution can be obtained, just as it can for the nondebonded imperfect beam, but with $K_x^{(k)} = 0$ and $K_z^{(k)} = 0$.

Finally, note that we have taken $K_z^{(k)} \to \infty$ in the numerical results to avoid the possibility of material penetration phenomenon.[1,2] In practice, if $K_z^{(k)}$ is finite, then the lower surface of the $(k+1)$th layer and the upper surface of the kth layer may be in contact, especially in a vibrating beam, making the problem nonlinear. In this case, it is generally impossible to derive the exact elasticity solution.

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Shell Theory Accuracy with Regard to Initial Postbuckling Behavior of Cylindrical Shell

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Introduction

MANY studies concerned with the postbuckling behavior of cylindrical shells have been published in the past four decades. These studies were motivated by the fact that the behavior of cylindrical shells under buckling is characterized by the limit point rather than by the bifurcation point. Accordingly, the behavior is sensitive to initial imperfection; thus, a specific parameter in the relevant reduction factor, namely, the knockdown factor, acquires extreme importance. The factor is closely dependent on the postbuckling characteristic behavior.

When the behavior of structures characterized by the limit point is investigated, two main formulations exist:

The first is the quantitative approach, which consists in tracing the entire nonlinear equilibrium paths with emphasis on the level and direction of change of the stiffness during loading, as was done, for example, by Sheinman and Simitses[1] and Simitses et al.[2] (also the review paper of Simitses[3]). Under this approach, the complete nonlinear behavior is realized for a given imperfection shape and amplitude. It is extremely complicated and entails a heavy computational effort, and worst of all, it cannot cover all cases because each new configuration (of the geometry and/or of the imperfection) has to be reanalyzed from the beginning.

The second formulation is the qualitative approach, which consists in the parametric study of the shell in terms of its sensitivity to imperfection and its rating according to the postbuckling stiffness ratio, given by the initial change of stiffness slopes right at the bifurcation point. Koiter[4] was the first to show that the imperfection sensitivity of a structure is closely related to its initial postbuckling behavior, identified by the so-called Koiter $\boldsymbol{b}$ parameter. This was established in many well-known research works, for example, see Arbocz and Hol,[5] Budiansky,[6] and the review papers of Hutchinson and Koiter[7] and of Simitses.[3]

Most of the research concerning cylindrical shells used the simplest Donnell-type theory.[8] Comparison of different shell theories

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in terms of the buckling behavior[9] and in terms of postbuckling behavior under the quantitative approach[10,11] shows pronounced discrepancies associated with the aspect ratio. Hence, it is desirable to examine the shell theories under the qualitative approach. This is attempted in the present Note, in which the imperfection sensitivity parameter (specifically, Koiter's $\boldsymbol{b}$ parameter) for an isotropic cylindrical shell under axial compression is compared according to the simplest (Donnell,[8]), accurate (Sanders[12]), and more accurate (Timoshenko[13]) theories.

The nonlinear equilibrium differential equations for the three theories are derived on the basis of the kinematic approach by the use of the displacement components as the unknown dependent variables. The asymptotic technique is used to convert the nonlinear equations into three linear sets. These equations are solved through expansion of the dependent variables in Fourier series in the circumferential direction and in finite differences in the axial direction. The Galerkin procedure is then used to minimize the error due to the truncated form of the series.

A special algorithm was developed and used for examination of the accuracy of each theory. The results show that the adopted shell theory plays an important role. The simplest (Donnell[8]) theory, which is commonly used, yields a higher sensitivity than the actual level, but on the conservative side.

Formulation

The strain–displacement relation of the reference surface $\bar{\varepsilon}$ and the change of curvature χ, under the Donnell[8] ($\delta_1=\delta_2=0$), Sanders[12] ($\delta_1=1$, $\delta_2=0$), and Timoshenko[13] ($\delta_1=\delta_2=1$) approaches are given by (see Sheinman and Goldfeld[9])

$$\{\bar{\varepsilon}\}=\begin{Bmatrix}\bar{\varepsilon}_{xx}\\ \bar{\varepsilon}_{\theta\theta}\\ \bar{\gamma}_{x\theta}\end{Bmatrix}=\begin{Bmatrix}u_{,x}+\dfrac{1}{2}w_{,x}^2+\delta_2\dfrac{v_{,x}^2}{2}\\ \dfrac{v_{,\theta}}{r}+\dfrac{w}{r}+\dfrac{w_{,\theta}^2}{2r^2}+\delta_1\left(\dfrac{v^2}{2r^2}-\dfrac{vw_{,\theta}}{r^2}\right)+\delta_2\dfrac{(v_{,\theta}+w)^2}{2r^2}\\ \dfrac{u_{,\theta}}{r}+v_{,x}+\dfrac{w_{,x}w_{,\theta}}{r}-\delta_1\left(\dfrac{vw_{,x}}{r}\right)+\delta_2\dfrac{v_{,x}v_{,\theta}}{r}\end{Bmatrix}$$

$$\{\chi\}=\begin{Bmatrix}\chi_{xx}\\ \chi_{\theta\theta}\\ 2\chi_{x\theta}\end{Bmatrix}=\begin{Bmatrix}-w_{,xx}\\ -\dfrac{w_{,\theta\theta}}{r^2}+\delta_1\dfrac{v_{,\theta}}{r^2}\\ -2\dfrac{w_{,x\theta}}{r}+\delta_1\dfrac{v_{,x}}{r}\end{Bmatrix}\tag{1}$$

where x and θ are the axial and circumferential cylindrical coordinates; r is the radius; and u, v, and w are the axial, circumferential, and normal displacement, respectively. Applying the variational principle, we obtain the following nonlinear equilibrium equations:

$$N_{xx,x}+N_{x\theta,\theta}/r+q_{xx}=0$$

$$N_{x\theta,x}+N_{\theta\theta,\theta}/r+\delta_1\Big[M_{\theta\theta,\theta}\big/r^2+M_{x\theta,x}/r+\big(N_{\theta\theta}\big/r^2\big)(w_{,\theta}-v)+(N_{x\theta}/r)w_{,x}\Big]+\delta_2\Big\{(N_{xx}v_{,x})_{,x}+[N_{\theta\theta}(v_{,\theta}+w)]_{,\theta}\big/r^2+(N_{x\theta}v_{,x})_{,\theta}/r+(N_{x\theta}v_{,\theta})_{,x}/r\Big\}+q_{\theta\theta}=0$$

$$M_{xx,xx}+2M_{x\theta,x\theta}/r+M_{\theta\theta,\theta\theta}\big/r^2-N_{\theta\theta}/r+(N_{xx}w_{,x})_{,x}+(N_{\theta\theta}w_{,\theta})_{,\theta}\big/r^2+(N_{x\theta}w_{,x})_{,\theta}/r+(N_{x\theta}w_{,\theta})_{,x}/r-\delta_1[(N_{\theta\theta}v)_{,\theta}\big/r^2+(N_{x\theta}v)_{,x}/r]-\delta_2\big(N_{\theta\theta}\big/r^2\big)(v_{,\theta}+w)+q_{zz}=0\tag{2}$$

with the following boundary conditions:

$$\begin{aligned}&u \quad\text{or}\quad N_{xx}\\ &v \quad\text{or}\quad N_{x\theta}+\delta_1(M_{x\theta}/r)+\delta_2(N_{xx}v_{,x}+N_{x\theta}v_{,\theta}/r)\\ &w \quad\text{or}\quad M_{xx,x}+2M_{x\theta,\theta}/r+N_{xx}w_{,x}+N_{x\theta}w_{,\theta}/r-\delta_1(N_{x\theta}v/r)\\ &w_{,x} \quad\text{or}\quad M_{xx}\end{aligned}\tag{3}$$

N_{ij} and M_{ij} are the membrane forces and bending moments and q_{xx}, $q_{\theta\theta}$, and q_{zz} are the external distributed loading in the x, θ, and normal z directions, respectively.

The asymptotic procedure yields separate sets of equations for the prebuckling, buckling, and initial postbuckling stages. The first two states have already been treated by Sheinman and Goldfeld.[9] The third state can be identified by

$$\lambda/\lambda_c=1+b\xi^2\tag{4}$$

with λ being the load parameter, λ_c the classical buckling load, and ξ the perturbation parameter. As for Koiter's sensitivity parameter $\boldsymbol{b}$, it is given by

$$\boldsymbol{b}=-\frac{(p_{211}+2p_{112})}{\lambda_c(p_{011}+2p_{110})}\tag{5}$$

where

$$\begin{aligned}p_{ijk}=\iint_{x\theta}\Bigg\{&N_{xx}^{(i)}\big\lfloor w_{,x}^{(j)}w_{,x}^{(k)}+\delta_2 v_{,x}^{(j)}v_{,x}^{(k)}\big\rfloor\\ &+\frac{N_{\theta\theta}^{(i)}}{r^2}\Big[w_{,\theta}^{(j)}w_{,\theta}^{(k)}+\delta_1\big(v^{(j)}v^{(k)}-v^{(j)}w_{,\theta}^{(k)}+v^{(k)}w_{,\theta}^{(j)}\big)\\ &+\delta_2\big(v_{,\theta}^{(j)}v_{,\theta}^{(k)}+w^{(j)}w^{(k)}+v_{,\theta}^{(j)}w^{(k)}+v_{,\theta}^{(k)}w^{(j)}\big)\Big]\\ &+\frac{N_{x\theta}^{(i)}}{r}\Big[w_{,x}^{(j)}w_{,\theta}^{(k)}+w_{,x}^{(k)}w_{,\theta}^{(j)}-\delta_1\big(v^{(j)}w_{,x}^{(k)}+v^{(k)}w_{,x}^{(j)}\big)\\ &+\delta_2\big(v_{,x}^{(j)}v_{,\theta}^{(k)}+v_{,x}^{(k)}v_{,\theta}^{(j)}\big)\Big]\Bigg\}\,\mathrm{d}x\,\mathrm{d}\theta\end{aligned}\tag{6}$$

where the superscripts (i), (j), and (k) denote the state in question, (0) for prebuckling, (1) for buckling, and (2) for postbuckling. A positive value of $\boldsymbol{b}$ indicates that the shell is insensitive, and a negative one indicates the sensitivity level, that is, the lower $\boldsymbol{b}$, the higher the sensitivity.

A special-purpose computer code was written that covered the buckling and initial postbuckling behavior of any isotropic cylindrical shell.

Results and Discussion

An example of a clamped–clamped cylindrical shell under axisymmetric axial compression is examined. The data are as follows: elastic modulus $E=1.404\times10^{11}$ N/m^2, Poisson's ratio $=0.2$, thickness $h=0.0127$ m, radius $r=1.27$ m ($r/h=100$), with boundary conditions out-of-plane $w=w_{,x}=0$ for both ends, and in-plane $u=v=0$ at one end and $N_{x\theta}=0$, $N_{xx}=\bar{N}_{xx}$ at the other. ($\bar{N}_{xx}$ is the applied external axial compression.) The buckling load and the buckling mode that governs the initial post buckling behavior are examined first in Figs. 1 and 2. In Fig. 1, the computational points are indicated by symbols. The dimensional buckling loads are plotted against the circumferential wave number for several length-to-radius (ℓ/r) ratios. It is shown that the more accurate the theory is (Donnell → Sanders → Timoshenko) the smaller the buckling wave number and that the higher the ℓ/r ratio is the larger the discrepancy and minimum wave number changes. (For convenience, the curves are assumed to be continuous with respect to the circumferential wave number.) For $\ell/r>3$ Donnell's theory yields almost the same buckling load for all wave numbers, (Fig. 2), whereas Sanders's and, more obviously, Timoshenko's, show a sharp value of the critical wave number (the one that yields the lowest buckling load), which may reflect on the initial postbuckling behavior. The lowest buckling load and the $\boldsymbol{b}$ parameter are plotted against ℓ/r in Fig. 3. Here, the $\boldsymbol{b}$ values are obtained through normalization of the postbuckling displacement via the Gram–Smith orthogonalization procedure.

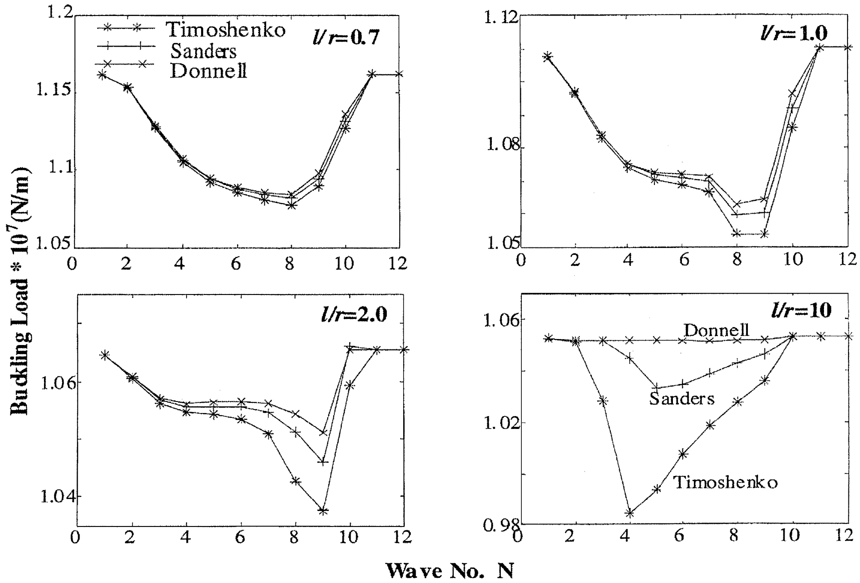

Fig. 1 Critical buckling circumferential mode according to the three theories.

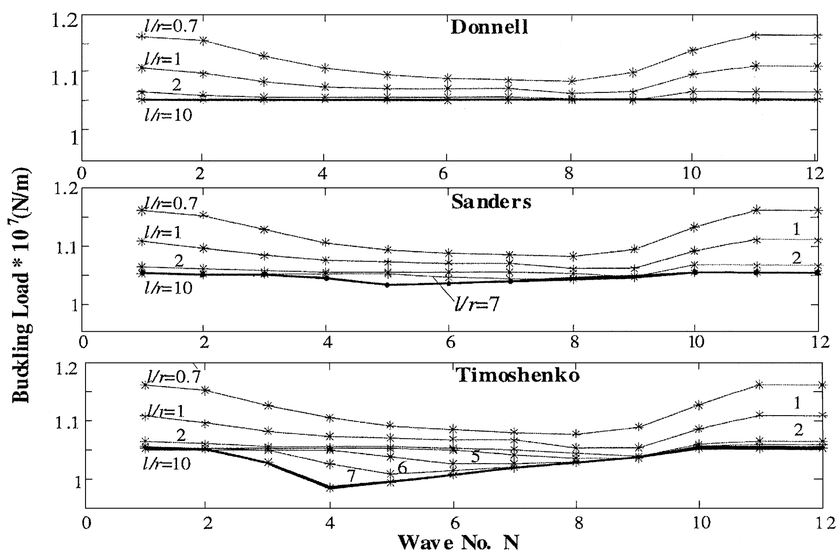

Fig. 2 Characteristic behavior of circumferential mode according to the three theories.

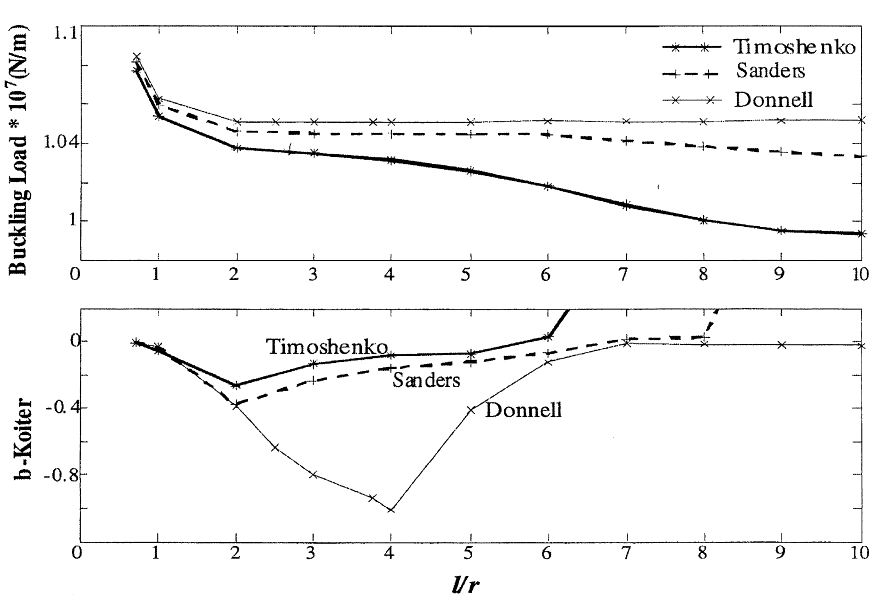

Fig. 3 Buckling load and sensitivity b parameter vs length-to-radius ratio.

Conclusions

The results lead to the following important conclusions:

1) The more accurate theory yields a lower buckling load.

2) The characteristic natural behavior of the $\boldsymbol{b}$ parameter changes significantly. The more accurate is the theory used, the lower is the sensitivity found. Furthermore, for certain values of ℓ/r (namely, $\ell/r > 5$ and $\ell/r > 7$ for Timoshenko and Sanders, respectively) the shell is totally insensitive. According to Donnell, the highest sensitivity corresponds to $\ell/r = 4$, according to Sanders and Timoshenko to $\ell/r = 2$. To sum up, the Donnell-type equations yield a higher sensitivity than the actual level, but on the conservative side.

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